Dynamical Study of the Singularities of Gravity in the Presence of Nonminimally Coupled Scalar Fields

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We investigate the dynamics of Einstein equations in the vicinity of the two recently described types of singularity of anisotropic and homogeneous cosmological models described by the action

$$S = \int d^4x \sqrt{-g}\{F(\phi)R - \partial_a\phi\partial^a\phi - 2V(\phi)\},$$

with general $F(\phi)$ and $V(\phi)$. The dynamical nature of each singularity is elucidated, and we show that both are, in general, dynamically unavoidable, reinforcing the unstable character of previous isotropic and homogeneous cosmological results obtained for the conformal coupling case.

KEY WORDS: singularities; gravity; scalar field.

1. INTRODUCTION

We have recently (Abramo *et al.*, 2003) studied the singularities of homogeneous and anisotropic solutions of cosmological models described by the action

$$S = \int d^4x \sqrt{-g}\{F(\phi)R - \partial_a\phi\partial^a\phi - 2V(\phi)\}, \tag{1}$$

with general $F(\phi)$ and $V(\phi)$. Such singularities appeared in the study of the robustness of previously considered (Gunzig *et al.*, 2001) homogeneous and isotropic solutions of cosmological models governed by (1) with $F(\phi) = 1 - \frac{1}{6}\phi^2$, corresponding to the so-called conformal coupling, and $V(\phi) = \frac{m}{2}\phi^2 - \frac{\Omega}{4}\phi^4$. These homogeneous and isotropic solutions present some novel and interesting dynamical behaviors such as superinflation regimes, a possible avoidance of big-bang and big-crunch singularities through classical birth of the universe from empty Minkowski

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space, spontaneous entry into and exit from inflation, and a cosmological history suitable in principle for describing quintessence. The appearance of the singularities implies that these results are not robust; they are radically changed, even for small disturbances in initial conditions and in the model itself. We have shown that the singularities are, essentially, of two types. The first one corresponds to the hypersurfaces $F(\phi) = 0$. It is not present in the isotropic case, and it implies that all previous homogeneous and isotropic solutions passing from the $F(\phi) > 0$ to the $F(\phi) < 0$ region are extremely unstable against anisotropic perturbations. The second type of singularity corresponds to $F_1(\phi) = 0$, with

$$F_1(\phi) = F(\phi) + \frac{3}{2}(F'(\phi))^2, \tag{2}$$

and it is present even for the homogeneous and isotropic cases. Although for small deviations from the conformal coupling the latter singularities are typically very far from the region of interest, in the general case they can alter qualitatively the global dynamics of the model due to restrictions that they impose on the phase space. Again, the persistence of some of our previously described results, in particular the ones concerning heteroclinic and homoclinic solutions, are challenged.

Both kinds of singularities have already been described before. To the best of our knowledge, Starobinski (1981) was the first to identify the singularity corresponding to the hypersurfaces $F(\phi) = 0$, for the case of conformally coupled anisotropic solutions. Futamase and coworkers (Futamase *et al.*, 1981; Futamase and Maeda, 1989) identified both singularities in the context of chaotic inflation in $F(\phi) = 1 - \xi\phi^2$ theories (see also Bertolami, 1987; Deser, 1984; Hosatani, 1985). The first singularity is always present for $\xi > 0$ and the second one for $0 < \xi < 1/6$. Our conclusions were, however, more general since we treated the case of general $F(\phi)$ and our results were based on the analysis of true geometrical invariants. Our main result is that the system governed by (1) is *generically* singular on both hypersurfaces $F(\phi) = 0$ and $F_1(\phi) = 0$. Here, *generically* means that it is possible to construct nonsingular models if one fine-tunes $F(\phi)$ and $V(\phi)$, as we have shown in the literature (Abramo *et al.*, 2003). The physical relevance of such a fine-tuned model is still unclear.

As was shown in the literature (Abramo *et al.*, 2003), one can advance that there are some geometrically special regions in the phase space of the model in question by a very simple analysis of the equations derived from the action (1). They are the Klein–Gordon equation

$$\Box\phi - V'(\phi) + \frac{1}{2}F'(\phi)R = 0, \tag{3}$$

and the Einstein equations

$$F(\phi)G_{ab} = (1 + F''(\phi))\partial_a\phi\partial_b\phi - \frac{1}{2}g_{ab}[(1 + 2F''(\phi))\partial_c\phi\partial^c\phi + 2V(\phi)]$$

$$- F'(\phi)(g_{ab}\Box\phi - \nabla_a\phi\nabla_b\phi). \tag{4}$$

We consider now the simplest anisotropic homogeneous cosmological model, the Bianchi type I, whose spatially flat metric is given by

$$ds^2 = -dt^2 + a^2(t)dx^2 + b^2(t)dy^2 + c^2(t)dz^2. \tag{5}$$

The dynamically relevant quantities here are

$$H_1 = \frac{\dot{a}}{a}, \qquad H_2 - \frac{\dot{b}}{b}, \quad \text{and} \quad H_3 = \frac{\dot{c}}{c}. \tag{6}$$

For such a metric and a homogeneous scalar field $\phi = \phi(t)$, after using the Klein–Gordon Eq. (3), Eq. (4) can be written as

$$F(\phi)G_{00} = \frac{1}{2}\dot{\phi}^2 + V(\phi) - F'(\phi)(H_1 + H_2 + H_3)\dot{\phi}, \tag{7}$$

$$\frac{1}{a^2}F(\phi)G_{11} = \frac{1+2F''(\phi)}{2}\dot{\phi}^2 - V(\phi) - F'(\phi)\left(H_1\dot{\phi} + V'(\phi) - \frac{F'(\phi)}{2}R\right), \tag{8}$$

$$\frac{1}{b^2}F(\phi)G_{22} = \frac{1+2F''(\phi)}{2}\dot{\phi}^2 - V(\phi) - F'(\phi)\left(H_2\dot{\phi} + V'(\phi) - \frac{F'(\phi)}{2}R\right), \tag{9}$$

$$\frac{1}{c^2}F(\phi)G_{33} = \frac{1+2F''(\phi)}{2}\dot{\phi}^2 - V(\phi) - F'(\phi)\left(H_2\dot{\phi} + V'(\phi) - \frac{F'(\phi)}{2}R\right), \tag{10}$$

It is quite simple to show that Eqs. (8)–(10) are not compatible, in general, on the hypersurface $F(\phi) = 0$. Subtracting (9) and (10) from (8) we have, on such hypersurface, respectively,

$$F'(\phi)(H_1 - H_2)\dot{\phi} = 0 \quad \text{and} \quad F'(\phi)(H_1 - H_3)\dot{\phi} = 0. \tag{11}$$

Hence, they cannot be fulfilled in general for anisotropic metrics. As it was shown, this indeed corresponds to a geometrical singularity that cannot be prevented in general by requiring that $F'(\phi) = 0$ or $\dot{\phi} = 0$ on the hypersurface.

As to the second singularity we have, after taking the trace of the Einstein equations, that

$$R = R(\phi, \dot{\phi}) = \frac{1}{F_1(\phi)}(4V(\phi) + 3V'(\phi)F'(\phi) - (1 - F''(\phi))\dot{\phi}^2). \tag{12}$$

Inserting Eq. (12) in the Klein–Gordon Eq. (3), one can see that it contains terms that are singular for $F_1(\phi) = 0$. Again, as we will see, this corresponds to an unmovable geometrical singularity, and it cannot be eliminated, in general, by demanding that $F'(\phi) = 0$ on the hypersurface $F_1(\phi) = 0$. On both the hypersurfaces $F(\phi) = 0$ and $F_1(\phi) = 0$ the Cauchy problem is ill-posed, since one cannot choose general initial conditions.

The hypersurfaces $F(\phi) = 0$ and $F_1(\phi) = 0$ also prevent the global definition of an Einstein frame for the action (1), defined by the transformations

$$\tilde{g}_{ab} = F(\phi)g_{ab}, \tag{13}$$

$$\left(\frac{d\tilde{\phi}}{d\phi}\right)^2 = \frac{F_1(\phi)}{2F(\phi)^2}. \tag{14}$$

It is well known that in the Einstein frame the Cauchy problem is well posed. Again, the impossibility of defining a global Einstein frame shed some doubts about the general Cauchy problem. Moreover, the standard perturbation theory for helicity-2 and helicity-0 excitations, derived directly from Eqs. (13) to (14), fails on both hypersurfaces (Esposito-Farese and Polarski, 2001).

The question to be addressed in the following sections is the dynamical behavior of the Eqs. (3) and (7)–(10) in the vicinity of the two hypersurfaces corresponding to $F(\phi) = 0$ and $F_1(\phi) = 0$. As we will see, both hypersurfaces are dynamically unavoidable, meaning that they have an attractive neighborhood, excluding definitively the possibility that these singularities are hidden by some dynamical barrier that would prevent the solutions to reach them. Whenever a solution enter in the attractive neighborhood, it will unavoidably reach the singular hypersurface.

2. NONCONSERVATIVE SYSTEMS AND THE DIVERGENCE THEOREM

For Hamiltonian systems, as a consequence of Liouville theorem, phase space volumes are preserved under the system time evolution. That means that if one chooses an initial closed hypersurface S_0 in the phase space and let each point of S_0 evolve in time according to the system equations, the closed hypersurface S_0 will evolve to another closed hypersurface S_t at some latter time t, and the volumes V of the region enclosed by S_0 and S_t are exactly the same, $V(0) = V(t)$. This is a characteristic of conservative systems. The system given by Eqs. (3) and (7)–(10) is not conservative. By choosing the set of coordinates $(\phi, \psi\ p, q, r)$ (with $\psi = \dot{\phi}\ p = H_1 + H_2 + H_3, q = H_1 - H_2,$ and $r = H_1 - H_3$), for the phase space $\mathcal{P}$, Eqs. (3) and (7)–(10) can be cast in the form

$$\dot{\phi}, \dot{\psi}, \dot{p}, \dot{q}, \dot{r} = \vec{W}(\phi, \psi, p, q, r). \tag{15}$$

For the metric (5), we have the following identities

$$G_{00} = H_1 H_2 + H_2 H_3 + H_1 H_3,$$

$$G_{11} = a^2 \left(\dot{H}_1 + H_1(H_1 + H_2 + H_3) - \frac{1}{2}R \right),$$

$$G_{22} = b^2 \left(\dot{H}_2 + H_2(H_1 + H_2 + H_3) - \frac{1}{2}R \right),$$

$$G_{33} = c^2 \left(\dot{H}_3 + H_3(H_1 + H_2 + H_3) - \frac{1}{2}R \right),$$

$$R = 2\left(\dot{H}_1 + \dot{H}_2 + \dot{H}_3 + H_1^2 + H_2^2 + H_3^2 + H_1 H_2 + H_2 H_3 + H_1 H_3 \right). \tag{16}$$

Using them, the components of $\vec{W}$ can be explicitly computed from Eqs. (7) to (10),

$$W_\phi = \psi$$

$$W_\psi = -p\psi - V'(\phi) + \frac{1}{2} F'(\phi) R(\phi, \psi),$$

$$W_p = -\left[(F(\phi) + 2F'(\phi)^2)p^2 + \frac{3}{2}(1 + 2F''(\phi))\dot{\phi}^2 - 3V(\phi) - 3F'(\phi)V'(\phi) \right.$$

$$\left. - p\dot{\phi} + (F(\phi) + F'(\phi)^2)(q^2 + r^2 - qr) \right] / (2F_1(\phi))$$

$$W_q = -\left(p + \frac{F'(\phi)}{F(\phi)} \psi \right) q,$$

$$W_r = -\left(p + \frac{F'(\phi)}{F(\phi)} \psi \right) r. \tag{17}$$

The divergence theorem assures us that the volume V of a closed hypersurface S_t of $\mathcal{P}$ evolves in time as

$$\dot{V}(t) = \int_{S_t} (\text{div}\,\vec{W})\,d\,\text{vol}, \tag{18}$$

where the integral is performed in the region enclosed by S_t. The divergence of the vector field $\vec{W}$ determines, therefore, how fast a volume of a closed hypersurface of $\mathcal{P}$ is expanded (div $\vec{W} > 0$) or contracted (div $\vec{W} < 0$). For conservative systems, one has div $\vec{W} = 0$. A straightforward calculation here give us

$$\text{div}\,\vec{W} = -p - 2\left(p + \frac{F'(\phi)}{F(\phi)} \psi \right)$$

$$- \frac{(F(\phi) + 2F'(\phi)^2)p + \left(F'(\phi)(1 + F''(\phi)) - \frac{1}{2}F'(\phi)\right)\psi}{F_1(\phi)} \tag{19}$$

It is clear from Eq. (19) that our system suffers violent contractions and/or expansions in the neighborhood of the hypersurfaces $F(\phi) = 0$ and $F_1(\phi) = 0$. Let us consider each of them separately since they lead to different kinds of singularity.

3. PHASE SPACE CONTRACTION AND EXPANSION NEAR $F(\phi) = 0$

Homogeneous and isotropic solutions, for whose $q = r = 0$, are known to be perfectly regular on $F(\phi) = 0$ (Gunzig *et al.*, 2001), in contrast to Eq. (19) that presents unequivocally a divergence on this hypersurface. A closer analysis of the vector field $\vec{W}$ (17) reveals that the divergent contractions and expansions near $F(\phi) = 0$ are associated to the directions q and r, namely the quantities that measure the anisotropy of the solution. The other directions of the flux defined by W

are regular on the hypersurface $F(\phi)$. That means that if a solution is perpendicular to the divergent directions q and r, solutions for which $q = r = 0$, i.e. isotropic solutions, it will evolve without suffering any violent contraction or expansion in the other directions of $\mathcal{P}$. Since W_q and W_r are proportional to q and r, respectively, an initially isotropic solution $q(0) = r(0) = 0$ remains isotropic for all latter t, $q(t) = r(t) = 0$. We can say that isotropic solutions are orthogonal to the divergent fluxes. However, any amount, no matters how small, of anisotropy (nonvanishing q or r) will break the orthogonality and the solution will prove the divergent directions, being strong contracted or expanded. This is the dynamical origin of the instabilities of anisotropic solutions near $F(\phi) = 0$.

Let us suppose now that $F'(\phi) \neq 0$ on the hypersurface $F(\phi) = 0$ (if $F'(\phi)$ vanishes on the hypersurface $F'(\phi) = 0$, then by Eq. (2) both hypersurfaces $F(\phi) = 0$ and $F_1(\phi) = 0$ coincide). The corresponding pole on ϕ_0 ($F(\phi_0) = 0$) in the volume integral (18) will have as numerator the factor $-2F'(\phi_0)\psi$, implying that the flux defined by Eq. (17) passes from a catastrophic contraction to a catastrophic expansion as one passes by ϕ_0. Since $W_\phi = \psi$ (see Eq. (17)), any solution approaching the hypersurface $F(\phi) = 0$ with $\psi \neq 0$ (we excluded from the analysis the possibility of having fixed points on $F(\phi) = 0$, for these cases, of course, it makes no sense to talk about "crossing" $F(\phi) = 0$) will cross it and, hence, prove the divergent phases of contraction and expansion.

In the expanding "side" of the hypersurface $F(\phi) = 0$, q and r diverges as $\phi \to \phi_0$, and the system will be unavoidably driven toward a spacetime singularity (Abramo *et al.*, 2003), as we can conclude by considering, for instance, the Kretschman invariant $I = R_{abcd}R^{abcd}$, which for the metric (5) is given by

$$I = 4\left(\left(\dot{H}_1 + H_1^2\right)^2 + \left(\dot{H}_2 + H_2^2\right)^2 + \left(\dot{H}_3 + H_3^2\right)^2 + H_1^2 H_2^2 + H_1^2 H_3^2 + H_2^2 H_3^2\right).$$

$$(20)$$

The invariant I is the sum of non-negative terms. Moreover, any divergence of the variables H_1, H_2, H_3, or of their time derivatives, would suppose a divergence in I, characterizing a real geometrical singularity. Since the relation between the variables p, q, r, and H_1, H_2, H_3 is linear, any divergence in p, q, r or of their time derivative, will suppose a divergence in I.

4. PHASE SPACE CONTRACTION AND EXPANSION NEAR $F_1(\phi) = 0$

The hypersurface $F_1(\phi) = 0$, as the previously one considered in the last section, also separates regions of catastrophic contraction and expansion in the phase space $\mathcal{P}$. However, in contrast to the previous one, this hypersurface leads to singularities even for homogeneous and isotropic solutions. The divergent directions now are p and ψ, and there are no solutions of (15) orthogonal to them, since, in contrast to the q and r directions, for which W_q and W_r are respectively

proportional to q and r, W_ψ and W_p do not vanish for $\psi = 0$ and $p = 0$. In this case, no solution can escape from crossing $F_1(\phi) = 0$. In the expanding side of $F_1(\phi) = 0$, $p \to \infty$ as $\phi \to \phi_1$ ($F_1(\phi) = 0$), implying the divergence of the invariant (20).

5. FINAL REMARKS

The singularities described in the precedent section imply that the model presented in the literature (Gunzig *et al.*, 2001; Saa *et al.*, 2001) is not robust, since its main conclusions were a consequence of very especial initial conditions, i.e. they are valid only for solutions orthogonal (namely the isotropic ones) to the divergently expanding directions q and r. For instance, all homogeneous and isotropic solutions crossing the $F(\phi) = 0$ hypersurface are extremely unstable against anisotropic perturbations. Any deviation from perfect isotropy (expressed by nonvanishing q and r variables) for these solutions, however small, will lead catastrophically to a geometrical singularity. Many of the novel dynamical behaviors presented in the literature (Gunzig *et al.*, 2001; Saa *et al.*, 2001) depend on these solutions. This is the case, for instance, of some solutions exhibiting superinflation regimes. The heteroclinic and homoclinic solutions identified in the literature (Gunzig *et al.*, 2001; Saa *et al.*, 2001) can cross the $F(\phi) = 0$ hypersurface and, hence, they also suffer the same instability against anisotropic perturbations. The homoclinic solutions were considered as candidates to describe a nonsingular cosmological history, with the big-bang singularity being avoided through a classical birth of the universe from empty Minkowski space. Apart from $F(\phi) = 0$ singularities, these solutions are also affected by the singularities of the type $F_1(\phi) = 0$. Suppose that the conformal coupling is disturbed by a very small negative term $F(\phi) = 1 - (\frac{1}{6} - \epsilon)\phi^2$. The $F_1(\phi) = 0$ singularities will be near the $\phi = \pm 1/\sqrt{\epsilon}$ hypersurfaces. Although they are located far from the $F(\phi) = 0$ regions, they alter the global structure of the phase-space. In this case, they restrict the existence of homoclinics, rendering a nonsingular cosmological history more improbable.

The singularities do not affect the conclusions obtained by considering solutions inside the $F(\phi) > 0$ region. The asymptotic solutions presented in the literature (Saa *et al.*, 2001), for instance, are still valid. The conclusion that for large t the dynamics of any solution (inside $F(\phi) > 0$) tends to an infinite diluted matter-dominated universe remains valid. Moreover, for small anisotropic deviations (q and r small in comparison with p), the solutions inside $F(\phi) > 0$, for large t, approach exponentially isotropic matter-dominated universe.

A singularity-free model can be constructed by demanding a well-behaved div $\vec{W}$ on both hypersurfaces. This can be achieved (Abramo *et al.*, 2003) by requiring $F(\phi_0) = F'(\phi_0) = 0$, and by choosing a $V(\phi)$ that goes to 0 at a proper rate when $\phi \to \phi_0$. Moreover $F_1(\phi)$ must have no other zeros than the ones of $F(\phi)$. Models for which $F(\phi) = \zeta\phi^{2n}$ and $V(\phi) = \alpha\phi^{2(2n-1)} +$ high order terms, for instance,

fulfill these requirements. However, such a highly fine-tuned class of model is of no physical interest here, since it does not contain $F(\phi) > 0$ and $F(\phi) < 0$ regions and consequently has no solution for which the effective gravitational constant G_{eff} changes it sign along the cosmological history. The stability of such solutions were the starting point of the analyses of the pioneering work (Starobinski, 1981) and of the present one as well.

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